PRESIDENT'S OFFICE

REGIONAL ADMINISTRATION AND LOCAL GOVERMENT

DAR ES SALAAM REGION

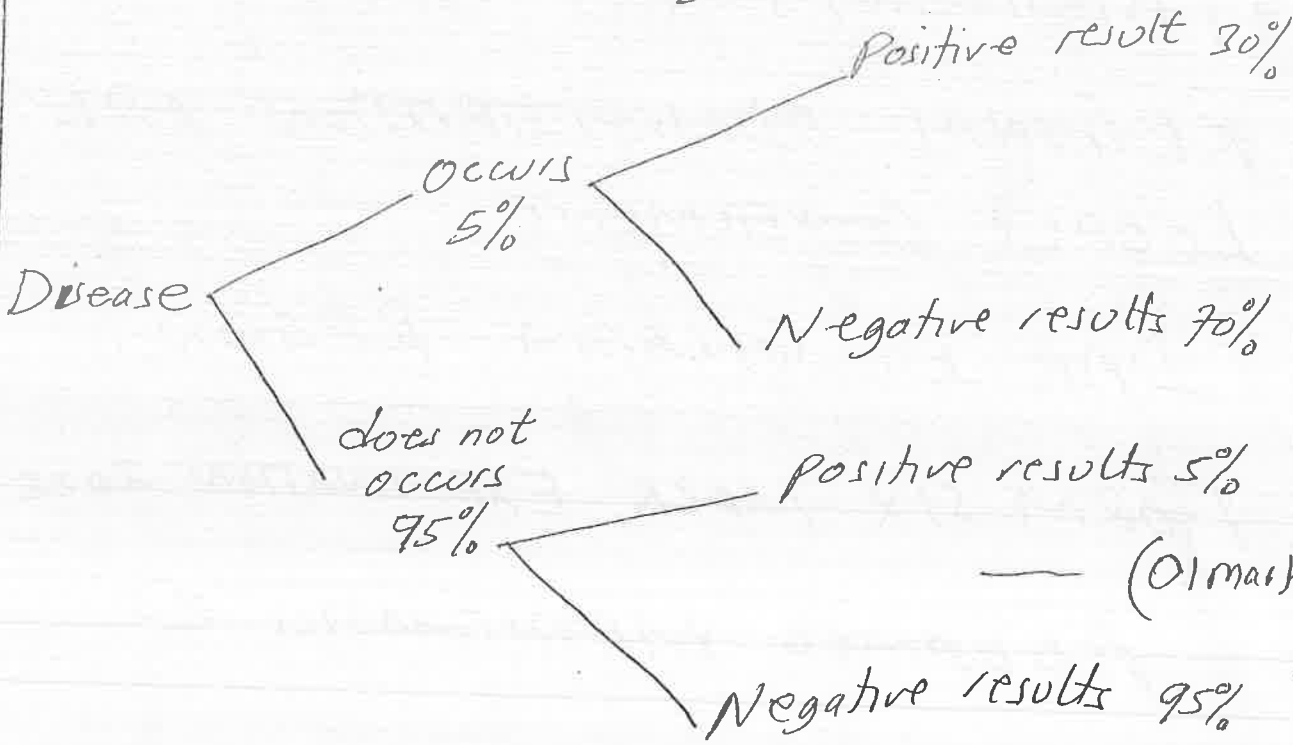
FORM SIX MOCK EXAMINATION 2025

ADVANCED MATHEMATICS 2

MARKING SCHEME

SECTION A

1 ⓐ A tree diagram



— (01 marks)

let D = disease occurs
D' = disease does not occurs
S = Positive results — (00½ Mark)
S' = Negative results

$P(D) = 0.05$, $P(D') = 0.95$
$P(S/D) = 0.3$, $P(S/D') = 0.05$ — (01 mark)

(i) By Multiplication rule
The probability that a randomly selected individual does not have the disease but gives a positive results in the screening test is

$P(D' \cap S) = P(S/D') \times P(D')$
$= 0.05 \times 0.95$ — (01 mark)

1 0.0475 or 4.75% — (00½ marks)

(ii) $P(D|S) = \dfrac{P(D\cap S)}{P(S)} = \dfrac{0.3\times0.05}{(0.3\times0.05)+(0.05\times0.95)}$

$= 0.24$ — (02 mark)

(b) As X is a random variable

$$\sum_{all\ x} P(X=x) = 1$$

x	0	1	2	3	...
$P(X=x)$	a	$\frac{3}{4}a$	$\frac{9}{16}a$	$\frac{27}{64}a$	

— (00½ mark)

$$\sum_{All\ x} P(X=x) = a + \frac{3}{4}a + \frac{9}{16}a + \frac{27}{64}a + \cdots$$ — (00½ mark)

$$= a\left(1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots\right)$$

$$= a\left(\frac{G_1}{1-r}\right)$$

$$= a\left(\frac{1}{1-3/4}\right)$$

$1 = 4a$ — (01 mark)

$\underline{a = 1/4}$

(c)

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Consider a probability distribution table below

x	0	1	2	3	4	...
$p(x)$	$e^{-\lambda}$	$\lambda e^{-\lambda}$	$\frac{e^{-\lambda}\lambda^2}{2!}$	$\frac{e^{-\lambda}\lambda^3}{3!}$	$\frac{e^{-\lambda}\lambda^4}{4!}$	...

for discrete $\sum_{x=0}^{\infty} P(x) = 1$

$$LHS = \sum_{x=0}^{\infty} P(x)$$

(01 mark)

$$= e^{-\lambda} + \lambda e^{-\lambda} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^3}{3!} + \frac{e^{-\lambda}\lambda^4}{4!} + \cdots$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots\right)$$

(01 mark)

$$= e^{-\lambda}(e^{\lambda}) = e^{-\lambda+\lambda} = e^0 = 1$$

shown

(ii) From $E(x) = \sum_{x=0}^{\infty} x P(x)$

$$E(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E(x) = \sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda \lambda^{x-1}}{x(x-1)!}$$

$$E(x) = \lambda \sum \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$ (01 mark)

$$E(x) = \lambda(1) = \lambda$$

Since $\sum_{x=a}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = 1$

$\therefore E(x) = \lambda$ shown — (01 mark)

(iii) From $Var(x) = E(x^2) - (E(x))^2$

Consider $E(x^2) = \sum x^2 P(x)$ but

$$x^2 = x(x-1) + x$$

$$E(x^2) = \sum_{x=0}^{\infty} (x(x-1)+x)P(x)$$

$$E(x^2) = \sum_{x=0}^{\infty} x(x-1)P(x) + \sum_{x=0}^{\infty} xP(x)$$

$$E(x^2) = \sum_{x=0}^{\infty} x(x-1)P(x) + \lambda$$ — (01 mark)

$$E(x^2) = \sum_{x=0}^{\infty} \frac{x(x-1)\, e^{-\lambda}\lambda^2 \times \lambda^{x-2}}{x(x-1)(x-2)!} + \lambda$$

$$E(x^2) = \lambda^2 \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} + \lambda$$

$$E(x^2) = \lambda^2(1) + \lambda$$

Since $\sum \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = 1$

$$E(x^2) = \lambda^2 + \lambda$$ — (01 mark)

(6/06)

$Var(x) = \lambda^2 + \lambda - \lambda^2$

$Var(x) = \lambda$ shown ——— (01 Marks)

2 (d) $[(p \to q) \wedge (\sim q \to (\sim p \wedge q)] \to \sim r$

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \to q$	$\sim p \wedge q$	a	b	c
T	T	T	F	F	F	T	T	T	T	F
T	T	F	F	F	T	T	T	T	T	T
T	F	T	F	T	F	F	T	T	F	T
T	F	F	F	T	T	T	T	T	F	T
F	T	T	T	F	F	T	T	T	T	F
F	T	F	T	F	T	T	T	T	T	T
F	F	T	T	T	F	T	T	T	T	F
F	F	F	T	T	T	T	T	T	T	T

$a = \sim q \to (\sim p \wedge q)$

$b = (p \to q) \wedge a$

$c = b \to \sim r$

Is not valid since last column contain T and F

2 (a) We have $[(p\to q)\wedge\sim q]\to\sim p$

$[(p\to q)\wedge\sim q]\to\sim p$ --- Given

$\sim((p\to q)\wedge\sim q)\vee\sim p$ --- implication law

$\sim(\sim q\wedge(\sim p\vee q))]\vee\sim p$ --- commutative law

$\sim((\sim q\wedge\sim p)\vee(\sim q\wedge q))]\vee\sim p$ --- distributive law

$\sim((\sim q\wedge\sim p)\vee F)\vee\sim p$ --- Negation law

$\sim(\sim q\wedge\sim p)\vee\sim p$ --- Identity law (04 mark)

$(q\vee p)\vee\sim p$ --- De morgan's law

$q\vee(p\vee\sim p)$ --- Associative law

$q\vee T$ --- Negation law

T --- Identity law

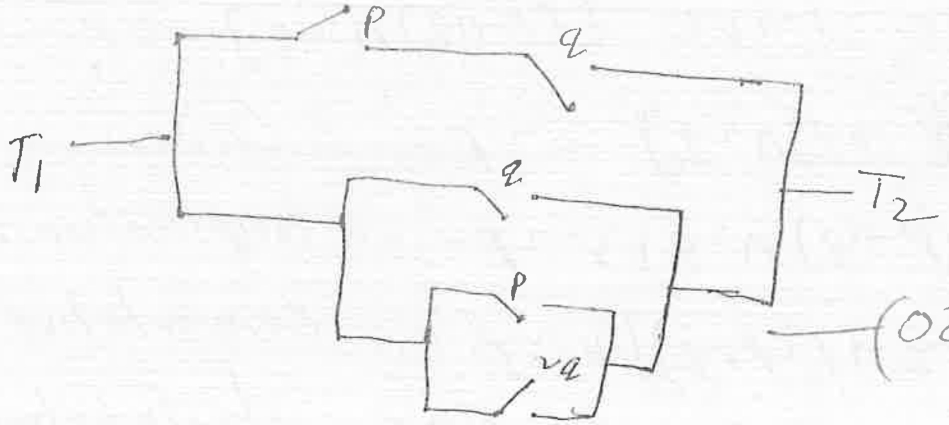
$\therefore [(p\to q)\wedge\sim q]\to\sim p\equiv T$ this Tautology

(b) We have $(p\to q)\to[q\to(p\to\sim q)]$

$\sim(\sim p\vee q)\vee(\sim q\vee(\sim p\vee\sim q))$

$(p\wedge\sim q)\vee(\sim q\vee(\sim p\vee\sim q))$ — (01 mark)

this electric network



(02mark)

(c) $p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r)$

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$\equiv$

(03mark)

(d) let

$p \equiv$ Water is boiled

$q \equiv$ free from germs.

$r \equiv$ people will be infected

Statement

$[(p \to q) \wedge (\sim q \to \sim r) \wedge (\sim p \wedge q)] \to \sim r$

its truth table

						a	b	c	d		
p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \to q$	$\sim q \to \sim r$	$a \wedge b$	$\sim p \wedge q$	$c \wedge d$	$(c \wedge d) \to \sim r$
T	T	T	F	F	F	T	T	T	F	F	T
T	T	F	F	F	T	T	T	T	F	F	T
T	F	T	F	T	F	F	F	F	F	F	T
T	F	F	F	T	T	F	T	F	F	F	F
F	T	T	T	F	F	T	T	T	T	T	F
F	T	F	T	F	T	T	T	T	T	T	T
F	F	T	T	T	F	T	F	F	F	F	T
F	F	F	T	T	T	T	T	T	F	F	T
						00½	00½	00½	00½	00½	00½

Therefore the argument is not (04marks)
valid since the last column
contain both T and F (01marks)

03 (a) (i)

$c = 3a - b$

$d = 2a - 10b$ θ between a and $b = 60°$

$c \cdot d = (3a - b) \cdot (2a - 10b)$

$c \cdot d = 6|a|^2 - 30a \cdot b - 2b \cdot a + 10|b|^2$

$c \cdot d = 6|a|^2 - 30|a||b|\cos\theta - 2|b||a|\cos\theta + 10|b|^2$

$c \cdot d = 6k^2 - 30k^2\cos 60° - 2k^2\cos 60° + 10(k)^2$

$c \cdot d = 6k^2 - 32k^2\cos 60° + 10k^2$

$c \cdot d = 16k^2 - 32k^2\left(\frac{1}{2}\right)$ 04

$c \cdot d = 16k^2 - 16k^2$

$c \cdot d = 0$ Hence shown that

c and d are perpendicula.

(ii) Given $i + j + k$ & $2i + 4j - 5k$ and $xi + 2j + 3k$

Sum of $= 2i + 4j - 5k + xi + 2j + 3k$

Sum $= (x+2)i + 6j - 2k$

unit vector $= \dfrac{(x+2)i + 6j - 2k}{\sqrt{(x+2)^2 + 36 + 4}}$ 06

$x = 1$

$$A = \frac{1}{2} b h.$$

$$A = \frac{1}{2} |(c - a)| h.$$

but

$$\sin\theta = \frac{h}{|b-a|} \Rightarrow h = |b-a| \sin\theta$$

$$A = \frac{1}{2} |c-a||b-a| \sin\theta.$$

$$A = \frac{1}{2} |(c-a) \times (b-a)|$$

05

$$A = \frac{1}{2} |c \times b - c \times a - a \times b + a \times a|$$

$$A = \frac{1}{2} |c \times b - c \times a - a \times b|$$

$$A = \frac{1}{2} |c \times b + a \times c + b \times a|$$ Hence shown

03 (c) $d = (1, 0, 3) - (2, -2, -3)$

$$d = (-1, 2, 6)$$

$$F_T = f_r (\hat{F})$$

$$F_T = \sqrt{7^2 + 4^2 + (-4)^2} \left(\frac{3i - 6j + 2k}{\sqrt{9 + 36 + 4}} \right)$$

$$F_T = 9 \left(\frac{3i - 6j + 2k}{7} \right)$$

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4 (a) $\dfrac{z+i}{iz+2} = \dfrac{x+iy+i}{i(x+iy)+2}$

$= \dfrac{x+i(y+1)}{2-y+ix}$ — — (03 marks)

$= \dfrac{x+i(y+1)}{2-y+ix} \times \dfrac{(2-y)-ix}{(2-y)-ix}$

$= \dfrac{x(2-y)-ix^2+i(y+1)(2-y)+x(y+1)}{(2-y)^2+x^2}$

$\dfrac{z+i}{iz+2} = \dfrac{3x}{(2-y)^2+x^2} - \dfrac{x^2+y^2-y-2}{(2-y)^2+x^2}i$

(b) $z = \cos\theta + i\sin\theta$

$\dfrac{1}{z} = \cos\theta - i\sin\theta$

$z - \dfrac{1}{z} = 2i\sin\theta$

$(2i\sin\theta)^6 = \left(z - \dfrac{1}{z}\right)^6$ (04 marks)

$-64\sin^6\theta = z^6 - 6z^4 + 15z^2 - 20 + \dfrac{15}{z^2} - \dfrac{6}{z^4} + \dfrac{1}{z^6}$

$-64\sin^6\theta = z^6 + \dfrac{1}{z^6} - 6\left(z^4 + \dfrac{1}{z^4}\right) + 15\left(z^2 + \dfrac{1}{z^2}\right) - 20$

$$\sin^6\theta = \frac{-1}{64}\left(2(\cos 6\theta) - 12\cos 4\theta + 30\cos 2\theta - 20\right)$$

$$\sin^6\theta = \frac{1}{32}\left[10 - 15\cos t + 6\cos 2t - \cos 3t\right]$$

(c) $1 + 2\sqrt{3}z^4 + 2iz^4 = 0$

$$(2\sqrt{3} + 2i)z^4 = -1$$

$$z^4 = \frac{-1}{2\sqrt{3}+2i}$$

$$z^4 = \frac{-1}{2\sqrt{3}+2i} \times \frac{2\sqrt{3}-2i}{2\sqrt{3}-2i}$$

$$z^4 = \frac{2\sqrt{3}-2i}{12+4} = \frac{\sqrt{3}}{8} - \frac{1}{8}i$$ (04 marks)

let $z = 1(\cos\theta + i\sin\theta)$

$w = \sqrt{3}/8 - \frac{1}{8}i$ into polar form

$$|w| = \sqrt{\left(\frac{\sqrt{3}}{8}\right)^2 + \left(\frac{-1}{8}\right)^2} = 1/4$$

$$\text{Arg}\left(\sqrt{3}/8 - \frac{1}{8}i\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\pi/6$$

$$w = \frac{1}{4}\left(\cos(-\pi/6) + i\sin(-\pi/6)\right)$$

$$W = \frac{1}{4}\left(\cos(-\pi/6) + i\sin(-\pi/6)\right)$$

$$r^4(\cos 4\theta + i\sin 4\theta) = \frac{1}{4}\left(\cos\left(\frac{-\pi}{6}\right) + i\sin(-\pi/6)\right)$$

By comparing

$$r^4 = 1/4 \qquad r = \frac{\sqrt{2}}{2}$$

$$\cos 4\theta = \cos(-\pi/6)$$

$$4\theta = 2\pi k - \pi/6$$

$$\theta = \frac{2\pi k - \pi/6}{4}$$

$$\theta = \frac{2\pi k - \pi}{24}$$

$$Z_{k+1} = \frac{\sqrt{2}}{2} e^{\left(\frac{2\pi k - \pi}{24}\right)i} \text{ where}$$

$$k = 0, 1, 2, 3 \ldots$$

(d) $\quad Arg\left(\frac{z-1}{z+1}\right) \le \pi/4$

(04 marks)

$$Arg(z-1) - Arg(z+1) \le \pi/4$$

$$Arg(x+iy-1) - Arg(x+iy+1) \le \pi/4$$

$$Arg(x-1+iy) - Arg(x+1+iy) \le \pi/4$$

$$\tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) \le \pi/4$$

$$\tan^{-1}\left(\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)}\right) \leq \pi/4$$

$$\frac{y(x+1)-y(x-1)}{(x-1)\cdot(x+1)+y^2} \leq \tan \pi/4$$

$$\frac{xy+y-xy+y}{x^2-1+y^2} \leq 1$$

$$2y \leq x^2-1+y^2$$

$$x^2+y^2-2y+1 \geq 0$$

$$x^2+y^2-2y \geq 1$$

$$x^2+y^2-2y+1 \geq 1+1$$

$$x^2+(y-1)^2 \geq 2$$

this is circle with centre $(0,1)$ and radius $\sqrt{2}$ units

16mc

SECTION B

5 (a) $\cos\theta = \frac{\sqrt{2}}{2}$

$\sec\theta = \sqrt{2}$

$\operatorname{cosec}\theta = \sqrt{2}$

$$= \frac{5\cos\theta + \tan\theta - \frac{1}{2}\operatorname{cosec}\theta}{3\cot\theta + \sec\theta - 2\operatorname{cosec}\theta}$$

$$= \frac{\frac{5\sqrt{2}}{2} + 1 - \frac{\sqrt{2}}{2}}{3 + \sqrt{2} - 2\sqrt{2}}$$

$$= \frac{2\sqrt{2}+1}{3-\sqrt{2}}$$

(~~04~~ 05 marks)

$$= \frac{2\sqrt{2}+1}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \sqrt{2}+1$$

$$\frac{5\cos\theta + \tan\theta - \frac{1}{2}\operatorname{cosec}\theta}{3\cot\theta + \sec\theta - 2\operatorname{cosec}\theta} = \sqrt{2}+1$$

(b) let $A = 2\tan^{-1}x$

$x = \tan A/2$

$\sin(2\tan^{-1}x) + \tan(\tan^{-1}x) = \sin A + \tan A$

$\sin(2\tan^{-1}x) + \tan(\tan^{-1}x) = \frac{\sin A}{\cos A}(\cos A + 1)$

But $\tan A = \frac{2\tan A/2}{1-\tan^2 A/2}$

$$\cos A = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

$$\sin(2\tan^{-1}x)+\tan(\tan^{-1}x) = \left(\frac{2\tan A/2}{1-\tan^2 A/2}\right)\left(\frac{1-\tan A/2}{1+\tan^2 A/2}+1\right)$$

$$= \left(\frac{2X}{1-X^2}\right)\left(\frac{1-X^2}{1+X^2}+1\right)$$

(04marks)

$$= \frac{4X}{1-X4}$$

$$\therefore \sin(2\tan^{-1}x)+\tan(\tan^{-1}x) = \frac{4X}{1-X4}$$

(c) $24\cos\left(\frac{\pi X}{3}\right)+10\sin\left(\frac{\pi x}{3}\right) = 13$

Consider

$$24\cos\left(\frac{\pi X}{3}\right)+10\sin\left(\frac{\pi X}{3}\right) = R\cos\left(\frac{\pi X}{3}-\alpha\right)$$

$$24\cos\left(\frac{\pi X}{3}\right)+10\cos\left(\frac{\pi X}{3}\right) = R\cos\frac{\pi x}{3}\cos\alpha + R\sin\frac{\pi X}{3}\sin\alpha$$

$$R\cos\frac{\pi x}{3}\cos\alpha = 24\cos\frac{\pi X}{3}$$

$$R\cos\alpha = 24 \text{ ——— (I)}$$

$$R\sin\frac{\pi x}{3}\sin\alpha = 10\sin\frac{\pi X}{3}$$

$$R\sin\alpha = 10 \text{ ——— (II)}$$

Square (i) and (ii) then add

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 10^2 + 24^2$$

$$R^2 = 676$$

$$R = 26$$

Divide eqn (ii) and (i)

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{10}{24}$$

$$\tan\alpha = 5/12$$ — (06 ~~05~~ marks)

$$\alpha = \tan^{-1}\left(\frac{5}{12}\right)$$

$$24\cos\frac{\pi x}{3} + 10\sin\frac{\pi x}{3} = 26\cos\left(\frac{\pi x}{3} - \tan^{-1}\left(\frac{5}{12}\right)\right)$$

$$26\cos\left(\frac{\pi x}{3} - \tan^{-1}\left(\frac{5}{12}\right)\right) = 13$$

$$\cos\left(\frac{\pi x}{3} - \tan^{-1}\left(\frac{5}{12}\right)\right) = \frac{1}{2}$$

$$\frac{\pi x}{3} - \tan^{-1}\left(\frac{5}{12}\right) = \cos\left(\frac{1}{2}\right)$$

$$\frac{\pi x}{3} - \tan^{-1}(5/12) = \pi/3$$

$$\frac{\pi x}{3} - \tan^{-1}\left(\frac{5}{12}\right) = 2\pi n \pm \pi/3$$

$$\frac{\pi x}{3} = 2\pi n \pm \frac{\pi}{3} + \tan^{-1}\left(\frac{5}{12}\right)$$

$$x = \frac{3}{\pi}\left(2\pi n \pm \frac{\pi}{3}\right) + \frac{3}{\pi}\tan^{-1}\left(\frac{5}{12}\right)$$

$$x = (6n \pm 1) + \frac{3}{\pi}\tan^{-1}\left(\frac{5}{12}\right)$$

Hence shown

(d) (1) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\theta = \tan \pi/_8$$

$$\tan\left(2\left(\frac{\pi}{8}\right)\right) = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\tan \pi/_4 = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$1 = \frac{2\tan\theta}{1-\tan^2\theta}$$

$1-\tan^2\theta = 2\tan\theta$ ... (~~04~~ 05 marks)

$$\tan^2\theta + 2\tan\theta - 1 = 0$$

$$\tan\theta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\tan\theta = \frac{-2 \pm \sqrt{4-4(-1)}}{2(1)}$$

$$\tan\theta = \frac{-2 \pm \sqrt{8}}{2}$$

$$\tan\theta = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$\tan\theta = \sqrt{2} - 1$$

but $\theta = \pi/8$

$$\tan \pi/8 = \sqrt{2} - 1$$

(ii) Also $\tan\left(\frac{3\pi}{8}\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{8}\right)$

$$\tan\left(\frac{3\pi}{8}\right) = \frac{\tan \pi/4 + \tan \pi/8}{1 - \tan\pi/4 \tan\pi/8}$$

$$\tan\left(\frac{3\pi}{8}\right) = \frac{1 + \sqrt{2} - 1}{1 - (\sqrt{2} - 1)}$$

04

... ($\cancel{0}$3 marks)

$$\tan\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2}}{2 + \sqrt{2}}$$

$$\tan\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2}}{2 - \sqrt{2}} \times \frac{2 + \sqrt{2}}{2 + \sqrt{2}}$$

$$\tan\left(\frac{3\pi}{8}\right) = \sqrt{2} + 1$$

6 (i) Given $a = \log_{12}18$ and $b = \log_{24}54$

$$a = \frac{\log 18}{\log 12} = \frac{\log(2 \times 3^2)}{\log(2^2 \times 3)}$$

$$= \frac{\log 2 + \log 3^2}{\log 2^2 + \log 3}$$

$$= \frac{\log 2 + 2\log 3}{2\log 2 + \log 3}$$

Divide by $\log 2$ to each term on RHS

$$= \frac{1 + 2\log_2 3}{2 + \log_2 3}$$

let $x = \log_2 3$ — (05marks)

$$a = \frac{1 + 2x}{2 + x}$$

Also $b = \log_{24} 54$

$$= \frac{\log 54}{\log 24} = \frac{\log(2 \times 3^2)}{\log(2^3 \times 3)}$$

$$= \frac{\log 2 + 2\log 3}{3\log 2 + \log 3}$$

(22/20)

$$b = \frac{\log 2 + 3\log 3}{3\log 2 + \log 3}$$

Divide by $\log 2$ to each term on RHS

$$= \frac{1 + 3\log_2 3}{3 + \log_2 3} \qquad \text{let } x = \log_2 3$$

$$b = \frac{1+3x}{3+x}$$

Since $x = \frac{2a-1}{2-a}$ then

$$b = \frac{1 + 3\left(\frac{2a-1}{2-a}\right)}{3 + \frac{2a-1}{2-a}} = \frac{2-a+6a-3}{6-3a+2a-1}$$

$$b = \frac{5a-1}{5-a}$$

$$b(5-a) = 5a-1$$

$$5b - ab = 5a - 1$$

$$5a - 5b + ab = 1$$

$\cancel{5ab}$

$$5a - 5b + ab = 1$$

$$5(a-b) + ab = 1$$

$$ab + 5(a-b) = 1$$ Hence shown.

(b) (i) $\frac{3x^2-x-1}{(x-1)^5}$

let $y=x-1$ $x=y+1$

$=\frac{3(y+1)^2-(y+1)-1}{y^5}$

$=\frac{3(y^2+2y+1)-y-1-1}{y^5}$

$=\frac{3y^2+6y+3-y-2}{y^5}$

$=\frac{3y^2+5y+1}{y^5}$

$=\frac{3y^2}{y^5}+\frac{5y}{y^5}+\frac{1}{y^5}$ --(03marks)

$=\frac{3}{y^3}+\frac{5}{y^4}+\frac{1}{y^5}$ but $y=x-1$

$=\frac{3}{(x-1)^3}+\frac{5}{(x-1)^4}+\frac{1}{(x-1)^5}$

$\frac{3x^2-x-1}{(x-1)^5}=\frac{3}{(x-1)^3}+\frac{5}{(x-1)^4}+\frac{1}{(x-1)^5}$

(11) Given $(x-1)^3 = 5$

$$\frac{3x^2-x-1}{(x-1)^5} = \frac{3}{5} + \frac{5}{(x-1)5} + \frac{1}{(x-1)^3(x-1)^2}$$

$$= \frac{3}{5} + \frac{1}{x-1} + \frac{1}{5(x-1)^2}$$

but $(x-1)^3 = 5$ — (03marks)

$$x-1 = \sqrt[3]{5} \text{ or } 5^{\frac{1}{3}}$$

$$= \frac{3}{5} + \frac{1}{\sqrt[3]{5}} + \frac{1}{5\sqrt[3]{5^2}}$$

$$= \frac{3}{5} + \frac{1}{5^{\frac{1}{3}}} + \frac{1}{25^{\frac{1}{3}}}$$

(c) We need to prove that

$$\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}$$ (05marks)

Express $\frac{1}{(3k-2)(3k+1)}$ in partial fractions

$$\frac{1}{(3k-2)(3k+1)} = \frac{A}{3k-2} + \frac{B}{3k+1}$$ — (01marks)

$$1 = A(3k+1) + B(3k-2)$$

$$A = \frac{1}{3} \text{ and } B = \frac{-1}{3}$$

$$\frac{1}{(3k-2)(3k+1)} = \frac{1}{3(3k-2)} - \frac{1}{3(3k+1)}$$

Therefore $\displaystyle\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \sum_{k=1}^{n} \frac{1}{3(3k-2)} - \frac{1}{3(3k+1)}$ (01 Mark)

Consider $\displaystyle\sum_{k=1}^{n} \frac{1}{3(3k-2)} - \frac{1}{3(3k+1)}$

When $k=1$ $\quad \frac{1}{3\times1} - \frac{1}{3\times4}$

When $k=2$ $\quad \frac{1}{3\times4} - \frac{1}{3\times7}$ (01 Mark)

When $k=3$ $\quad \frac{1}{3\times7} - \frac{1}{3\times10}$

⋮

When $k=n-2$ $\quad \frac{1}{3(3n-8)} - \frac{1}{3(3n-5)}$

$k=n-1$ $\quad \frac{1}{3(3n-5)} - \frac{1}{3(3n-2)}$

$k=n$ $\quad \frac{1}{3(3n-2)} - \frac{1}{3(3n+1)}$ (01 Mark)

Therefore

$$\sum_{k=1}^{n} \frac{1}{3(3k-2)} - \frac{1}{3(3k+1)} = \frac{1}{3\times1} - \frac{1}{3(3n+1)}$$

$$\sum_{K=1}^{n} \frac{1}{(3K-2)(3K+1)} = \frac{1}{3} - \frac{1}{3(3n+1)}$$

$$\sum_{K=1}^{n} \frac{1}{(3K-2)(3K+1)} = \frac{3n+1-1}{3(3n+1)}$$

$$\sum_{K=1}^{n} \frac{1}{(3K-2)(3K+1)} = \frac{3n}{3(3n+1)}$$ (01 marks)

$$\sum_{K=1}^{n} \frac{1}{(3K-2)(3K+1)} = \frac{n}{3n+1}$$

Hence proved.

(d) $X^3 + PX^2 + QX + 30 = 0$, Ratios 2:3:5

let roots are α, β and γ

But $\frac{\alpha}{K} = 2$ $\frac{\beta}{K} = 3$, $\frac{\gamma}{K} = 5$

$\alpha = 2K$, $\beta = 3K$ $\gamma = 5K$

$S = 2K + 3K + 5K = -P$

$10K = -P$ —— (1)

$$SP = (2k)(3k) + (2k)(5k) + (3k)(5k) = q$$

$$6k^2 + 10k^2 + 15k^2 = q$$

$$31k^2 = q \text{ ——— (ii)}$$

$$P = (2k)(3k)(5k) = -30$$

$$30k^3 = -30$$

$$k^3 = -1 \qquad (04\text{marks})$$

$$k = -1$$

Then using (i)

$$p = 10$$

$$q = 31(-1)^2$$

$$q = 31$$

The value of $p = 10$ and $q = 31$

7 (a) (i) $y = Ae^{bx+1}$

$y = Ae \cdot e^{bx}$

$\frac{dy}{dx} = Aebe^{bx}$

but $Aee^{bx} = y$

$dy/dx = b(Aee^{bx})$

$dy/dx = by$

$\Rightarrow \frac{1}{y}\, dy/dx = b$

$\frac{d}{dx}\left(\frac{1}{y}\frac{dy}{dx}\right) = b$ —— (03marks)

$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\frac{dy}{dx}\cdot\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 0$ or

$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$

(ii) $y = A\sin x + B\cos x$

$dy/dx = A\cos x - B\sin x$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

$$\frac{d^2y}{dx^2} = -(A\sin x + B\cos x)$$

but $y = A\sin x + B\cos x$

$$\frac{d^2y}{dx^2} = -y$$ — (03marks)

$$\frac{d^2y}{dx^2} + y = 0$$

(b) $(x^2+y^2) + 2xy\frac{dy}{dx} = 0$

Compare with

$M\,dx + N\,dy = 0$

$M = x^2+y^2$

$N = 2xy$ — (03marks)

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y$$

This DE is exact since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Now $\int (x^2+y^2)\,dx = \frac{1}{3}x^3 + xy^2$ — (I)

$\int 2xy\,dy = xy^2$ — (II)

Merging (i) and (ii) we get

$f(x) = \frac{1}{3}x^3 + xy^2$

Therefore solution is $\frac{1}{3}x^3 + xy^2 = c$

(c) $\frac{d^2y}{dx^2} + dy/dx - 8y = 0$ —— *

$y = e^{mx}$ —— (i)

$dy/dx = Me^{mx}$ —— (ii)

$\frac{d^2y}{dx^2} = M^2e^{mx}$ —— (iii)

Substituting (i) (ii) (iii) into *

$M^2e^{mx} + 2me^{mx} - 8e^{mx} = 0$

$M^2 + 2M - 8 = 0$ - (05marks)

Roots $X_1 = 2$ $X_2 = -4$ for $x = m$

If $M_1 = 2$ and $M_2 = -4$

$y = Ae^{M_1x} + Be^{M_2x}$

$y = Ae^{2x} + Be^{-4x}$

But $y(0) = 1$ and $y'(0) = 3$

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$y(0) = 1$

$y(0) = A + B$

$A + B = 1$ —— (1)

$$\frac{dy}{dx} = y'(x) = 2Ae^{2x} - 4Be^{-4x}$$

$y'(0) = 2A - 4B$

$2A - 4B = 3$ —— (11)

Solving (1) and (11) simultaneously

$A = 7/6$

$B = -1/6$

$$y = \frac{7}{6}e^{2x} - \frac{1}{6}e^{-4x}$$

$$y = \frac{1}{6}\left(7e^{2x} - e^{-4x}\right)$$

(d) Rate of cooling ∝ excess temperature over surrounding

$$\frac{d\theta}{dt} \propto (\theta - \theta_R)$$

$$\frac{d\theta}{dt} = -K(\theta - \theta_R)$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\theta - \theta_R} = \int_0^t -K\,dt$$

$$\int_{200}^{T} \frac{d\theta}{\theta-10} = \int_{0}^{t} -k\,dt$$

$$\ln(\theta-10)\Big|_{200}^{T} = -kt\Big|_{0}^{t}$$

$$\ln(T-10) - \ln(200-10) = -kt$$

$$\ln\left(\frac{T-10}{190}\right) = -kt$$

$$\frac{T-10}{190} = e^{-kt}$$

— (06marks)

$$T-10 = 190e^{-kt}$$

$$T = 10+190e^{-kt}$$

Where

$$K = \frac{1}{40}\ln\left(\frac{19}{9}\right)$$

Hence proved $T = 10+190e^{-kt}$

8 (a) Given $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$

$$dy/_{dx} = \frac{2a}{y}$$

$$M_1 = dy/_{dx} = \frac{2a}{y_1} \text{ at } (x_1, y_1)$$

from $M_1M_2 = -1$

$$\left(\frac{2a}{y_1}\right)M_2 = -1 \quad M_2 = \frac{-y_1}{2a}$$

Equation

$$M = \frac{y - y_1}{x - x_1}$$

$$\frac{-y_1}{2a} = \frac{y - y_1}{x - x_1} \quad \text{— (03marks)}$$

$$-y_1(x - x_1) = 2a(y - y_1)$$

$$-y_1x + y_1\cancel{x_1} = 2ay - 2ay_1$$

$$y_1\cancel{x} + 2ay - y_1x_1 - 2ay_1 = 0$$

$$y_1x + 2ay - y_1(x_1 + 2a) = 0$$

Hence shown.

(b) (i) let $y = x - c$ —— (1)

$$9x^2 + 16y^2 = 144 \text{ —— } (2)$$

e Substitute (1) into (2), we get

$9x^2+16(x-c)^2=144$

$9x^2+16(x^2-2xc+c^2)=144$

$9x^2+16x^2-32xc+16c^2=144$

$25x^2-32cx+16c^2-144=0$ —— (3)

for tangency $b^2=4ac$

$(-32c)^2=4(25)(16c^2-144)$

$1024c^2=1600c^2-14400$

$-576c^2=-14400$ ---- (03marks)

$c^2=25$

$c=\pm 5$

from (3)

~~$25(\pm 5)^2-$~~

$25x^2-32x(\pm 5)+16(\pm 5)^2-144=0$

$x=\frac{16}{5}$ $x=-\frac{16}{5}$

$y=x-c$

$y=x-\pm 5$

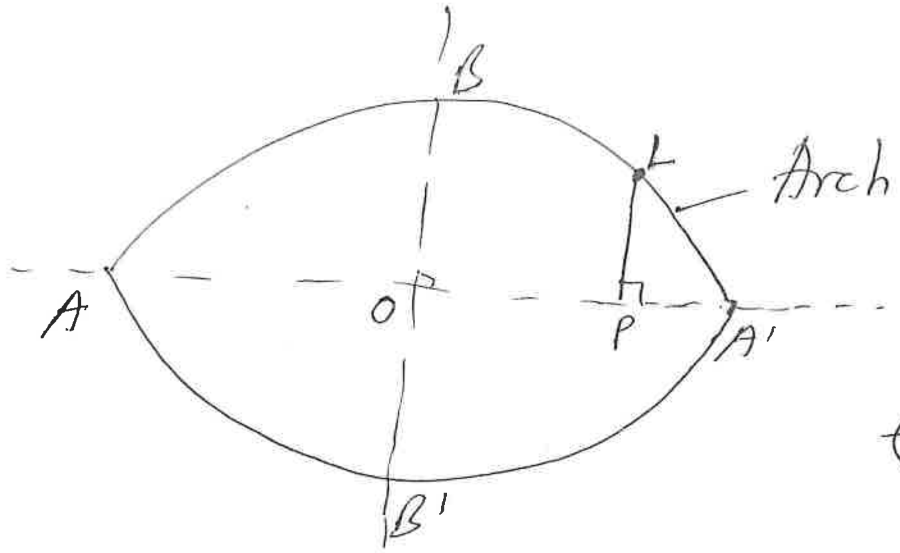
$y=-9/5$, $y=5/9$

Point of contact

$\left(\frac{16}{5}, \frac{-9}{5}\right)$ and $\left(\frac{-16}{5}, \frac{9}{5}\right)$

(11) Ellipse equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Consider



(04 marks)

Arch ABA' = Arch, Major axis = 8M

Minor axis = 3M

$AA' = 8M$ and OB = Height

$2a = 8M$ $OB = 3M$

$a = 4M$ $b = 3M$

then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{16} + \frac{y^2}{9} = 1$$

At point $P(x, y)$ is 1 from end

$a = OA' = 4$

$OP = OA' - PA'$

$OP = 4 - 1 = 3$

Thus coordinate at $P(3,0)$ and $L(3,y)$ using eqn

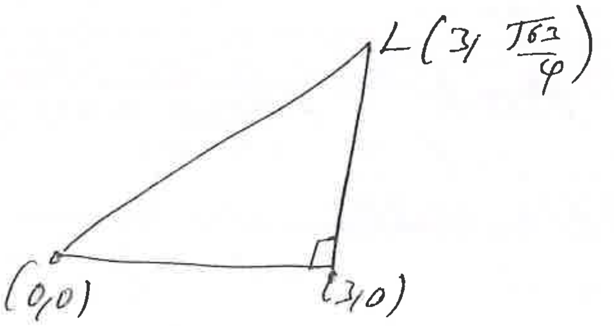
$$\frac{3^2}{16} + \frac{y^2}{9} = 1$$

$$y^2 = 63/16$$

$$y = \pm\frac{\sqrt{63}}{4}$$

the height of the arch $= \frac{\sqrt{63}}{4}$ M or 1.98M

Distance from centre



$$D = \sqrt{(3-0)^2 + \left(\frac{\sqrt{63}}{4} - 0\right)^2}$$

$$D = \sqrt{12.9375}$$

$$\underline{D = 3.5969\,M}$$

(c) $36y^2 - 25x^2 + 100x + 288y - 424 = 0$

$36(y^2 + 8y) - 25(x^2 - 4x) - 424 = 0$

$36\left(y^2 + 8y + \left(\frac{8}{2}\right)^2 - 16\right) - 25\left(x^2 - 4x + \left(\frac{4}{2}\right)^2 - 4\right) - 424 = 0$

$36[(y+4)^2 - 16) - 25((x-2)^2 - 4) - 424 = 0$

$36(y+4)^2 - 576 - 25(x-2)^2 + 100 - 424 = 0$

$36(y+4)^2 - 25(x-2)^2 = 900$

$$\frac{(y+4)^2}{25} - \frac{(x-2)^2}{36} = 1$$

Compare with

$$\frac{(y-q)^2}{b^2} - \frac{(x-p)^2}{a^2} = 1$$

-- (05 marks)

Centre $(p, q) = (2, -4)$

Vertices $= (p, q \pm b)$

$= (2, -4 \pm 5)$

Vertices $= (2, 1)$ and $(2, -9)$

Asymptotes $y - q = \pm \frac{b}{a}(x - p)$

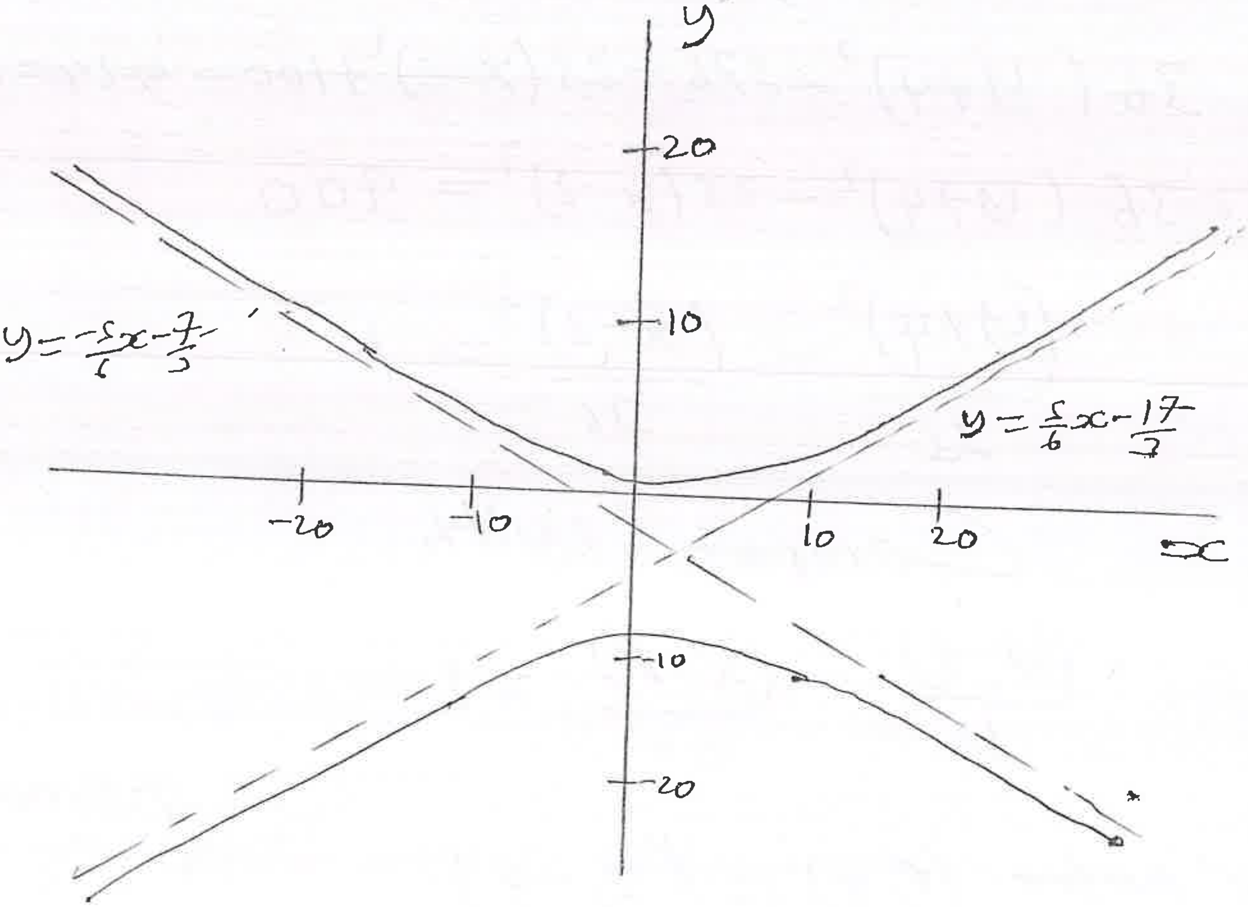
$y = \pm 5/6 \,(x - 2)$

36/30

Asymptotes $y = \frac{5}{6}x - \frac{17}{3}$ or

$y = \frac{-5}{6}x - \frac{7}{3}$

Its Graph



(d) (i) We have $\tan 2\theta = r^2$

Required to show $2xy = x^4 - y^4$

$$\frac{2\tan\theta}{1-\tan^2\theta} = r^2$$

But $\tan\theta = y/x$ and

$r^2 = x^2 + y^2$

(37/20)